X100/701

NATIONAL
QUALIFICATIONS
2007

TUESDAY, 15 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**

Marks

Answer all the questions.

1. Express the binomial expansion of $\left(x - \frac{2}{x}\right)^4$ in the form $ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$ for integers a, b, c, d and e. **4**

2. Obtain the derivative of each of the following functions:

(*a*) $f(x) = \exp(\sin 2x)$; **3**

(*b*) $y = 4^{(x^2+1)}$. **3**

3. Show that $z = 3 + 3i$ is a root of the equation $z^3 - 18z + 108 = 0$ and obtain the remaining roots of the equation. **4**

4. Express $\dfrac{2x^2 - 9x - 6}{x(x^2 - x - 6)}$ in partial fractions. **3**

Given that

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}\,dx = \ln\frac{m}{n},$$

determine values for the integers m and n. **3**

5. Matrices A and B are defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(*a*) Find the product AB. **2**

(*b*) Obtain the determinants of A and of AB. **2**

Hence, or otherwise, obtain an expression for det B. **1**

6. Find the Maclaurin series for $\cos x$ as far as the term in x^4. **2**

Deduce the Maclaurin series for $f(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4. **2**

Hence write down the first three non-zero terms of the series for $f(3x)$. **1**

Marks

7. Use the Euclidean algorithm to find integers p and q such that $599p + 53q = 1$. **4**

8. Obtain the general solution of the equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$. **6**

9. Show that $\sum_{r=1}^{n}(4-6r) = n - 3n^2$. **2**

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$. **1**

Show that $\sum_{r=q+1}^{2q}(4-6r) = q - 9q^2$. **2**

10. Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$. **5**

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{(1+x^2)^2}$ between $x = 0$ and $x = 1$ through 360° about the x-axis. Write down the volume of this solid. **1**

11. Given that $|z-2| = |z+i|$, where $z = x + iy$, show that $ax + by + c = 0$ for suitable values of a, b and c. **3**

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2| = |z+i|$. **1**

12. Prove by induction that for $a > 0$,

$$(1 + a)^n \geq 1 + na$$

for all positive integers n. **5**

13. A curve is defined by the parametric equations $x = \cos 2t$, $y = \sin 2t$, $0 < t < \frac{\pi}{2}$.

(*a*) Use parametric differentiation to find $\frac{dy}{dx}$.

Hence find the equation of the tangent when $t = \frac{\pi}{8}$. **5**

(*b*) Obtain an expression for $\frac{d^2y}{dx^2}$ and hence show that $\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = k$, where k is an integer. State the value of k. **5**

[Turn over for Questions 14 to 16 on *Page four*

14. A garden centre advertises young plants to be used as hedging.

After planting, the growth G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.

(*a*) Express G in terms of t and k. **4**

(*b*) Given that a plant grows 0·6 metres by the end of 5 years, find the value of k correct to 3 decimal places. **2**

(*c*) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? **2**

(*d*) Given that the initial height of the plants was 0·3 m, what is the likely long-term height of the plants? **2**

15. Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x = 2 + s,\ y = -s,\ z = 2 - s \qquad L_2 : x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

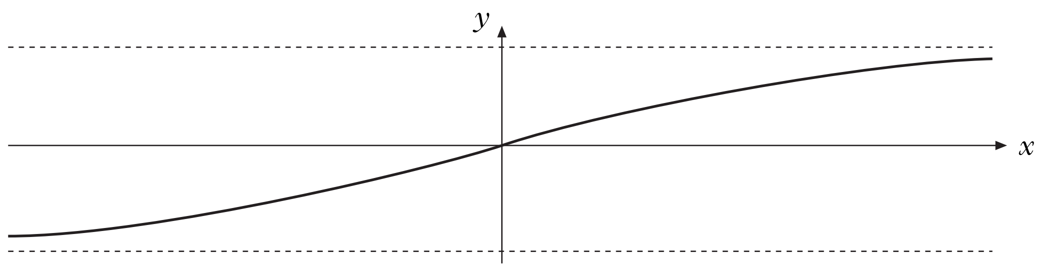
(*a*) Show that L_1 and L_2 do not intersect. **3**

(*b*) The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3. **3**

(*c*) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1. **3**

(*d*) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ. **1**

16.



(*a*) The diagram shows part of the graph of $f(x) = \tan^{-1} 2x$ and its asymptotes. State the equations of these asymptotes. **2**

(*b*) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$, $x = \frac{1}{2}$. **5**

(*c*) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x-axis and the lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$. **3**

[*END OF QUESTION PAPER*]